Unique continuation for $\bar{\partial}u = Vu$ at infinity

Yifei Pan and Yuan Zhang

Abstract

Motivated by Landis's conjecture on unique continuation at infinity for the Laplacian, we study the corresponding property for the Cauchy-Riemann operator. We prove that every weak solution of $\bar{\partial}u = Vu$ on a neighborhood of infinity, with $V \in L^\infty$, vanishes identically if it decays exponentially at a rate greater than $2\|V\|_{L^\infty}$. This conclusion is sharp both in the constant $2\|V\|_{L^\infty}$ and in the order of exponential decay required. More generally, we establish unique continuation at infinity for a broad class of radially decaying bounded potentials, with optimal decay rates determined by the decay of the potential. We also obtain related unique continuation results for L^2 potentials and for compactly supported potentials under weaker assumptions at infinity.

1 Introduction

Let u solve

$$-\Delta u = Vu \tag{1.1}$$

on $\mathbb{R}^d, d \geq 2$, where $V \in L^\infty(\mathbb{R}^d)$. In 1988, Landis [12] conjectured that if $|u(x)| \leq e^{-|x|^{1+\epsilon}}$ for some $\epsilon > 0$ whenever $|x| \gg 1$, then $u \equiv 0$. This may be viewed as a unique continuation property at infinity for solutions of (1.1) satisfying sufficiently rapid decay. The conjecture fails in general for complex-valued potentials. Indeed, Meshkov [16] constructed a nontrivial solution u_0 of (1.1) in $\mathbb{R}^2$ for some complex-valued potential $V \in L^\infty(\mathbb{R}^2)$ such that $|u_0(x)| \leq e^{-k|x|^{\frac{4}{3}}}$ for some $k > 0$ whenever $|x| \gg 1$. By contrast, the case of real-valued potentials has attracted considerable attention, particularly following the fundamental works of Bourgain–Kenig [2] and Kenig [10]. More recently, Logunov, Malinnikova, Nadirashvili and Nazarov [14] established the Landis conjecture for real-valued potentials in dimension two. In a recent preprint, Frank and Ivanisvili [6] constructed a nontrivial solution of (1.1) with a bounded real-valued potential that decays like $e^{-k|x|^{\frac{4}{3}}}$ in dimension three and higher, thereby providing a counterexample to the Landis conjecture in these dimensions.

The purpose of this paper is to investigate analogous unique continuation properties at infinity for the Cauchy-Riemann operator $\bar{\partial}$. More precisely, we consider complex-valued functions on a domain $\Omega \subset \mathbb{C}^n$ containing a neighborhood of infinity and satisfying

$$\bar{\partial}u = Vu \tag{1.2}$$

on Ω in the sense of distributions, where V is a measurable $(0,1)$-form on Ω. The main objective is to determine whether sufficiently rapid decay of a solution at infinity forces it to vanish identically, thereby establishing a unique continuation principle at infinity for first-order equations of Cauchy–Riemann type.

2020 Mathematics Subject Classification. Primary 32W05; Secondary 35B60.

Key words and phrases. Landis conjecture, Cauchy-Riemann operator, unique continuation, infinity.

In the case when the potential V is globally bounded, we first establish a quantitative lower bound for (normalized) solutions of (1.2) near infinity as follows.

Theorem 1.1. *Suppose $u \in C(\mathbb{C}^n)$ satisfies $u(0) = 1$ and*

$$|u(z)| \le e^{C_0|z|}, \qquad |z| \gg 1$$

for some $C_0 > 0$, and

$$\bar\partial u = Vu \quad on \quad \mathbb{C}^n$$

in the sense of distributions for some $(0,1)$-form $V \in L^\infty(\mathbb{C}^n)$. Then there exists a positive constant C depending only on n, C_0 and $\|V\|_{L^\infty(\mathbb{C}^n)}$ such that

$$\inf_{|z_0|=R} \sup_{|z-z_0|<1} |u(z)| \ge e^{-CR\ln R}, \quad R \gg 1.$$

The case $n = 1$ in Theorem 1.1 is due to Kenig, Silvestre, and Wang [11], who employed the three-circle theorem together with a scaling argument of Bourgain and Kenig [2]. Extending this approach to higher dimensions requires overcoming the compatibility obstruction for the $\bar\partial$-equation. The key observation is that the potential is $\bar\partial$-closed, making use of a result of Gong and Rosay [7]. Combined with sup-norm estimates of $\bar\partial$ and a higher-dimensional three-circle theorem, this yields the required quantitative lower bound.

The $R\ln R$ dependence in the exponent of Theorem 1.1 is sharp up to a multiplicative constant, as demonstrated in Remark 3.4 by a family of holomorphic functions for which this dependence cannot be improved. As a direct consequence, any solution to (1.2) that decays faster than $e^{-C|z|\ln|z|}$ must be trivial; see Corollary 3.8.

However, the rate in Corollary 3.8 is not optimal for the purpose of unique continuation. Indeed, Theorem 1.2 below improves upon this conclusion by showing that sufficiently rapid exponential decay already forces triviality. Moreover, rather than requiring the equation (1.2) on all of $\mathbb{C}^n$, we assume only that it holds on an exterior domain, for instance, $\mathbb{C}^n \setminus \overline{B_R}$, where B_R denotes the open ball centered at 0 with radius $R > 0$ in $\mathbb{C}^n$. The proof proceeds by slicing along a coordinate axis and then applying the similarity principle and the monotonicity of the winding number.

Theorem 1.2. *Suppose $u \in L^1_{loc}(\mathbb{C}^n \setminus \overline{B_R})$ satisfies*

$$\bar\partial u = Vu \quad on \quad \mathbb{C}^n \setminus \overline{B_R}$$

in the sense of distributions for some $(0,1)$-form $V \in L^\infty(\mathbb{C}^n \setminus \overline{B_R})$. If

$$|u(z)| \le e^{-C|z|}, \qquad |z| \gg 1 \tag{1.3}$$

for some $C > 2\|V\|_{L^\infty(\mathbb{C}^n\setminus\overline{B_R})}$, then u vanishes identically.

The linear exponential scale and the exponential constant in Theorem 1.2 are both optimal. Indeed, for any $c > 0$, consider the nontrivial smooth function $u = e^{-c|z|}$ on $\mathbb{C}^n \setminus \overline{B_1}$. Then $u \in L^1(\mathbb{C}^n \setminus \overline{B_1})$ and satisfies $\bar\partial u = Vu$ on $\mathbb{C}^n \setminus \overline{B_1}$, where $V = -\sum_{j=1}^n \frac{cz_j}{2|z|}\, d\bar z_j \in L^\infty(\mathbb{C}^n \setminus \overline{B_1})$ with $\|V\|_{L^\infty(\mathbb{C}^n\setminus\overline{B_1})} = \frac{c}{2}$, showing that the threshold $C > 2\|V\|_{L^\infty(\mathbb{C}^n\setminus\overline{B_R})}$ is sharp.

More generally, we establish a unified unique continuation result in Theorem 4.1 for a broad class of radially decaying bounded potentials. The decay required of the solution is determined explicitly by the decay of the potential at infinity, and the resulting threshold is sharp. This framework includes, for example, both the bounded-potential case of Theorem 1.2 and logarithmically decaying potentials; see Example 2.

We next consider the unique continuation problem at infinity for solutions satisfying the much weaker condition of L^2-flatness (1.4). The following theorem establishes unique continuation for $W^{1,2}$ solutions of (1.2) with L^2 potentials under this assumption. Here, for a positive integer k and $p \geq 1$, $W^{k,p}(\Omega)$ denotes the space of functions whose weak derivatives up to order k belong to $L^p(\Omega)$. The result also applies to vector-valued solutions $u = (u_1, \dots, u_N)^T$, with $u \in W^{k,p}(\Omega)$ understood componentwise. See also Corollary 5.3 for an application of Theorem 1.3 to the uniqueness of certain nonlinear equations involving the Laplacian under sufficiently rapid decay at infinity.

Theorem 1.3. *Suppose $u = (u_1, \dots, u_N)^T \in W^{1,2}(\mathbb{C}^n \setminus \overline{B_R})$ satisfies*

$$\bar\partial u = Vu \quad on \quad \mathbb{C}^n \setminus \overline{B_R}$$

for some $N \times N$ matrix-valued $(0,1)$-form $V \in L^2(\mathbb{C}^n \setminus \overline{B_R})$. Assume that u vanishes to infinite order at infinity in the L^2 sense, namely,

$$\lim_{r\to\infty} r^m \int_{|z|>r} |u(z)|^2 \, dv_z = 0 \qquad \text{for every } m > 0. \tag{1.4}$$

Then u vanishes identically.

When $n = 1$, as shown in the proof of Theorem 1.3, the problem at infinity can be transformed by inversion into a unique continuation problem at a finite point, with the L^2 norm of the potential preserved. In this case, L^2 is the critical integrability class: for every $p > 2$, there exist potentials in $L^p \setminus L^2$ for which unique continuation fails. In higher dimensions, however, the situation differs substantially from its finite-point counterpart. At a finite point, the critical exponent $2n$ is intrinsic to the problem, as reflected by the counterexamples in Remark 2.2; see also [21]. From the viewpoint of the proof, the reduction to one dimension is carried out by complex radial slicing, which leads naturally to this dimension-dependent integrability exponent. By contrast, at infinity one may use coordinate slicing together with Fubini's theorem, so that L^2 integrability of the potential is preserved on almost every slice. The one-dimensional unique continuation result can then be applied on almost every such slice, and the resulting vanishing is propagated to the full exterior domain by weak unique continuation. Thus the L^2 condition on the potential in Theorem 1.3 arises naturally from the one-dimensional sliced problem.

As shown by Example 4, unique continuation under the L^2-flatness assumption may fail for potentials outside L^2 at infinity. Nevertheless, Theorem 5.1 identifies an important class extending beyond L^2 class for which unique continuation remains valid, namely, potentials satisfying $V(z) = O\left(\frac{1}{|z|}\right)$ at infinity.

Furthermore, we consider compactly supported potentials on $\mathbb{C}^n$. In one complex dimension, Chirka and Rosay [4] established the following uniqueness result for the $\bar\partial$-equation: if $V \in L^\infty(\mathbb{C})$ has compact support and $u \in C^1(\mathbb{C})$ satisfies $\bar\partial u = Vu$ on $\mathbb{C}$ and $\lim_{z\to\infty} u = 0$, then $u \equiv 0$. Theorem 1.4 below extends this result to higher dimensions while relaxing the assumption $V \in L^\infty(\mathbb{C})$ to $V \in L^p(\mathbb{C}^n)$ for some $p > 2$.

Theorem 1.4. *Suppose $u \in L^2_{loc}(\mathbb{C}^n)$ satisfies*

$$\bar\partial u = Vu \quad on \quad \mathbb{C}^n$$

in the sense of distributions for some $(0,1)$-form $V \in L^p(\mathbb{C}^n), p > 2$. If V has compact support and

$$\lim_{z\to\infty} u(z) = 0,$$

then u vanishes identically.

Note that compact support of potentials alone is not sufficient for unique continuation. Indeed, Mandache's example shows that weak unique continuation may fail for compactly supported potentials below the L^2 integrability threshold; see Remark 2.2. We also recall that, in Meshkov's counterexample [16] to Landis's conjecture for (1.1) on $\mathbb{R}^2$, the complex-valued potential vanishes on a sequence of concentric annuli with radii tending to infinity. On the other hand, holomorphic functions clearly fall within the framework of Theorem 1.4 by taking $V \equiv 0$. When restricted in this special case, the conclusion also follows from the classical Liouville theorem, since boundedness already forces u to be constant. Consequently, if in addition $u(z) \to 0$ as $|z| \to \infty$, then necessarily $u \equiv 0$.

Finally, it is worth pointing out that the conclusions of Theorems 1.2 and 1.3, although stated on complements of closed balls, remain valid for any domain that is the complement of a compact set. This follows directly from the weak unique continuation property in Theorem 2.1. By contrast, the global nature of Theorems 1.1 and 1.4 is essential, as their conclusions may fail when the domain $\mathbb{C}^n$ is replaced by the complement of a compact set. This is already evident in one complex dimension. Indeed, for any $k_0 \in \mathbb{N}$, the nontrivial function $u(z) = z^{-k_0}, |z| > 1$ satisfies $\bar{\partial} u = 0 = Vu$ for $|z| > 1$ with $V \equiv 0$, and $\lim_{z\to\infty} u(z) = 0$, showing that Theorem 1.4 fails in this setting. See also Remark 3.5 for an example on the complement of the unit disc showing that Theorem 1.1 does not extend to exterior domains.

2 Preliminaries

In this section, we collect several preliminary results concerning unique continuation at a finite point, Sobolev properties for the $\bar{\partial}$ operator, one-dimensional inversion and coordinate slicing. These results will be used repeatedly in the proofs of the main theorems.

We first recall a unique continuation result for the $\bar{\partial}$ operator at a finite point under L^2-flatness. A function $u \in L^2_{loc}(\Omega)$ is said to vanish to infinite order, or to be L^2-flat, at a point $z_0 \in \Omega$ if

$$\lim_{r\to 0} r^{-m} \int_{|z-z_0|<r} |u(z)|^2 \, dv_z = 0 \qquad \text{for every } m > 0.$$

Theorem 2.1. *[18] [21] Let Ω be a domain in $\mathbb{C}^n$. Suppose $u = (u_1, \ldots, u_N)^T \in W^{1,2}_{loc}(\Omega)$ satisfies $|\bar{\partial} u| \le V|u|$ almost everywhere on Ω for some nonnegative measurable function V on Ω.*

1. *The weak unique continuation holds if $V \in L^2_{loc}(\Omega)$: if u vanishes in an open subset of Ω, then u vanishes identically.*

2. *The (strong) unique continuation holds if $V \in L^2_{loc}(\Omega)$ when $n = 1$, or if $V \in L^p_{loc}(\Omega)$ for some $p > 2n$ when $n \ge 2$: if u vanishes to infinite order in the L^2 sense at some $z_0 \in \Omega$, then u vanishes identically.*

3. *Suppose that $0 \in \Omega$ and $V \le \frac{C}{|z|}$ on Ω for some $C > 0$. Then the (strong) unique continuation property holds if either $N = 1$, or $N \ge 2$ and $C < \frac{1}{4}$: if u vanishes to infinite order in the L^2 sense at 0, then u vanishes identically.*

We note that every function $u = (u_1, \ldots, u_N)^T \in W^{1,1}_{loc}(\Omega)$ satisfying

$$|\bar{\partial} u| \le V|u| \quad \text{a.e. on} \quad \Omega$$

with $V \in L^p_{loc}(\Omega)$ can be viewed as a weak solution of a first order Schrödinger-type system for $\bar\partial$. Indeed, fix a measurable representative of u and let $Z = \{u = 0\}$. The set Z is well defined up to a null set. By the standard Stampacchia property that the weak gradient of a Sobolev function vanishes almost everywhere on each of its level sets, $\bar\partial u = 0$ almost everywhere on Z. Define the $N \times N$ matrix of $(0,1)$-forms $\mathcal{V} = (\mathcal{V}_{kj})$ with $\mathcal{V}_{kj} = 0$ on Z and $\mathcal{V}_{kj} := \frac{(\bar\partial u_k)\overline{u_j}}{|u|^2}$ otherwise. Then $\mathcal{V}_{kj} \in L^p_{loc}(\Omega)$, and

$$\bar\partial u = \mathcal{V} u \quad \text{a.e. on } \ \Omega.$$

On the other hand, if $f \in L^1_{loc}(\Omega)$ and $u \in W^{1,1}_{loc}(\Omega)$, then

$$\bar\partial u = f \qquad \text{on } \ \Omega$$

in the sense of distributions is equivalent to

$$\bar\partial u = f \qquad \text{a.e. on } \ \Omega.$$

Thus, results for the system $\bar\partial u = \mathcal{V} u$ apply equally to the differential inequality $|\bar\partial u| \le V|u|$ through the above reduction. We shall also freely pass between the almost-everywhere and distributional formulations whenever the relevant quantities belong to L^1_{loc}. For more results concerning unique continuation for the inequality $|\bar\partial u| \le V|u|$ at a finite point, see, for instance, [18, 20–22].

Remark 2.2. *The unique continuation fails in general in the following situations.*

1. *The weak unique continuation may fail if the potential does not belong to L^2_{loc}. Indeed, by an example of Mandache [15], for every $0 < p < 2$, there exist a nontrivial smooth function u on $\mathbb{C}$, supported in the unit disc, and a $(0,1)$-form $V \in L^p(\mathbb{C})$ such that $\bar\partial u = Vu$ on $\mathbb{C}$.*

2. *The strong unique continuation may fail if the potential does not belong to L^{2n}_{loc}. Indeed, for each $1 \le p < 2n$, choose $\epsilon \in (0, \frac{2n}{p} - 1)$. The function $u = e^{-\frac{1}{|z|^\epsilon}}$ extended by $u(0) = 0$ is smooth and vanishes to infinite order at 0. Moreover, $|\bar\partial u| \le V|u| := \frac{\epsilon}{2|z|^{\epsilon+1}}|u|$ on $B_1 \subset \mathbb{C}^n$. By the choice of ϵ, we have $V \in L^p(B_1)$.*

3. *The unique continuation property in Theorem 2.1 part 3 may also fail in general if $N \ge 2$ and C is sufficiently large, as shown by examples of the first author and Wolff [17], and of Alinhac and Baouendi [1]; see also [20, Example 5]. More recently, Chen, Fan and Tang [3] showed that $C = \frac{1}{4}$ is the sharp threshold for unique continuation. More precisely, they constructed a nontrivial smooth function that vanishes to infinite order at 0 and satisfies $|\bar\partial u| \le \frac{|u|}{4|z|}$ almost everywhere on $\mathbb{C}^n$.*

Regarding regularity, $\bar\partial$ is a first-order elliptic operator and improves local Sobolev $W^{k,p}, 1 < p < \infty$ regularity precisely by one order; see [21, Lemma 3.1]. Combining this property with a bootstrap argument yields the following regularity result for solutions of (1.2) with L^p potentials, $p > 2n$. Throughout the paper, whenever a Sobolev function admits a continuous representative. we always identify the Sobolev equivalence class with this representative. All subsequent pointwise statements concerning such functions are understood in this sense.

Lemma 2.3. *Let Ω be a domain in $\mathbb{C}^n$, $n \ge 1$, and let $u = (u_1, \ldots, u_N)^T$ satisfy*

$$\bar\partial u = Vu \qquad on \ \ \Omega$$

in the sense of distributions for some measurable matrix-valued $(0,1)$-form V.

1. *If $u \in L^2_{loc}(\Omega)$ and $V \in L^p_{loc}(\Omega)$ for some $2n < p < \infty$, then $u \in W^{1,p}_{loc}(\Omega)$.*

2. *If $u \in L^1_{loc}(\Omega)$ and $V \in L^\infty_{loc}(\Omega)$, then $u \in W^{1,q}_{loc}(\Omega)$ for every $1 \le q < \infty$.*

In particular, in either case we have $u \in C(\Omega)$.

Proof. We first prove part 1. By Hölder's inequality, $Vu \in L^{\frac{2p}{p+2}}_{loc}(\Omega)$ with $\frac{2p}{p+2} > 1$. Since the $\bar\partial$ operator improves local regularity precisely by one, it follows that $u \in W^{1,\frac{2p}{p+2}}_{loc}(\Omega)$. By the Sobolev embedding theorem, $u \in L^{\frac{2np}{np+2n-p}}_{loc}(\Omega)$. A standard bootstrap argument eventually gives $u \in L^{\tilde p}_{loc}(\Omega)$ for some $\tilde p > \frac{2np}{p-2n}$. Consequently, $Vu \in L^q_{loc}(\Omega)$ for some $q > 2n$. Elliptic regularity then gives $u \in W^{1,q}_{loc}(\Omega)$. Since $q > 2n$, the Sobolev embedding theorem also gives $u \in C(\Omega)$ and hence $Vu \in L^p_{loc}(\Omega)$. A final application of the local elliptic regularity of $\bar\partial$ yields $u \in W^{1,p}_{loc}(\Omega)$.

For part 2, since $V \in L^\infty_{loc}(\Omega)$, we have $Vu \in L^1_{loc}(\Omega)$. The local L^p estimates for $\bar\partial$ then gives $u \in L^{q_0}_{loc}(\Omega)$ for some $q_0 > 1$. Indeed, this follows from the L^p estimates of the solution operator T_q of $\bar\partial$ in [13, p. 86, Proposition 4.24], applied with $q = 0$ and $p = 1$, together with the fact that $u - T_0(Vu)$ is holomorphic. Consequently, $Vu \in L^{q_0}_{\mathrm{loc}}(\Omega)$. Applying the same bootstrap argument as above, we obtain $u \in W^{1,q}_{loc}(\Omega)$ for every $q < \infty$. Hence $u \in C(\Omega)$. □

Although the chain rule and/or product rule do not hold in general for Sobolev functions without additional assumptions, we identify a class of Sobolev functions for which they are valid. Since this result will be used repeatedly throughout the paper, we provide a proof below.

Lemma 2.4. *Let Ω be a domain in $\mathbb{R}^d, d \ge 2$. Suppose that $u \in W^{1,p}_{loc}(\Omega)$ and $\lambda \in W^{1,q}_{loc}(\Omega)$ for some $p \ge 1$ and $q > d$. Then $ue^\lambda \in W^{1,s}_{loc}(\Omega)$ with $s = \min\{p, q\}$, and*

$$\nabla(ue^\lambda) = e^\lambda \nabla u + ue^\lambda \nabla\lambda \quad \text{on} \quad \Omega \tag{2.1}$$

in the sense of distributions.

Proof. Since $\lambda \in W^{1,q}_{loc}(\Omega)$ for some $q > d$, the Sobolev embedding theorem implies that $\lambda \in C(\Omega)$. In particular, λ and e^λ are locally bounded in Ω. Thus by the Sobolev chain rule (see, for instance, [24, p. 48]), $e^\lambda \in W^{1,q}_{loc}(\Omega)$ and

$$\nabla e^\lambda = e^\lambda \nabla\lambda \quad \text{on} \quad \Omega \tag{2.2}$$

in the sense of distributions.

Let $\phi \in C^\infty_c(\Omega)$ and $U \Subset \Omega$ be a bounded smooth domain containing *supp* ϕ. Let $u_j (\in C^\infty(U)) \to u$ in $W^{1,p}(U)$ norm. Then

$$-\langle ue^\lambda, \nabla\phi\rangle = \lim_{j\to\infty} -\langle u_j e^\lambda, \nabla\phi\rangle = \lim_{j\to\infty} -\langle e^\lambda, (\nabla\phi)u_j\rangle = \lim_{j\to\infty} -\langle e^\lambda, \nabla(\phi u_j)\rangle + \lim_{j\to\infty}\langle e^\lambda, \phi\nabla u_j\rangle$$

Since $\phi u_j \in C^\infty_c(\Omega)$, one deduces from (2.2) that

$$\langle e^\lambda, \nabla(\phi u_j)\rangle = -\langle e^\lambda \nabla\lambda, \phi u_j\rangle.$$

Making use of the boundedness of e^λ on U and the fact that $u_j \to u$ in $L^{\frac{d}{d-1}}(U)$ norm due to the continuous embedding of $W^{1,p}(U)$ into $L^{\frac{d}{d-1}}(U)$, we have

$$\lim_{j\to\infty} -\langle e^\lambda, \nabla(\phi u_j)\rangle = \lim_{j\to\infty}\langle e^\lambda\nabla\lambda, \phi u_j\rangle = \langle ue^\lambda\nabla\lambda, \phi\rangle.$$

Similarly,

$$\lim_{j\to\infty}\langle e^{\lambda},\phi\nabla u_j\rangle = \langle e^{\lambda}\nabla u,\phi\rangle.$$

Altogether, we obtain the desired equality (2.1).

It remains to verify the claimed regularity. If $p \geq q$, then $p \geq q > d$, so the Sobolev embedding theorem gives $u \in L^{\infty}_{loc}(\Omega)$. Hence both terms on the right-hand side of (2.1) belong to $L^{q}_{loc}(\Omega)$. Suppose instead that $p < q$. The Sobolev embedding theorem gives $u \in L^{\frac{dp}{d-p}}_{loc}(\Omega)$ if $p < d$, and $u \in L^{\tilde{p}}_{loc}(\Omega)$ for all $\tilde{p} < \infty$ otherwise. Since $q > d$, it follows that $u \in L^{\frac{qp}{q-p}}_{loc}(\Omega)$. Hölder's inequality yields $u\nabla\lambda \in L^{p}_{loc}(\Omega)$. Since e^{λ} is locally bounded, both terms on the right-hand side of (2.1) belong to $L^{p}_{loc}(\Omega)$. Therefore, in either case, $ue^{\lambda} \in W^{1,s}_{loc}(\Omega)$. $\square$

We shall also use two preparatory lemmas. The first concerns the one-dimensional inversion that transforms a unique continuation problem at infinity into a local problem near a finite point, while the second is a coordinate-slicing lemma used to reduce higher-dimensional problems to one complex dimension.

Lemma 2.5. *Suppose $u = (u_1,\dots,u_N)^T \in W^{1,p}_{loc}(\mathbb{C}\setminus\overline{D_R})$, $p \geq 1$ satisfies*

$$\bar{\partial}u = Vu \quad a.e.\ on \quad \mathbb{C}\setminus\overline{D_R}$$

for some matrix-valued $(0,1)$-form $V \in L^{2}_{loc}(\mathbb{C}\setminus\overline{D_R})$. Define $v(w) := u(\frac{1}{w})$ and $W(w) := -\frac{V(\frac{1}{w})}{\bar{w}^2}, w \in D_{\frac{1}{R}}\setminus\{0\}$. Then $v \in W^{1,p}_{loc}(D_{\frac{1}{R}}\setminus\{0\})$ and $W \in L^{2}_{loc}(D_{\frac{1}{R}}\setminus\{0\})$, with

$$\bar{\partial}v = Wv \quad a.e.\ on \quad D_{\frac{1}{R}}\setminus\{0\}.$$

Proof. The inversion $F(w) = \frac{1}{w}$ is a smooth diffeomorphism from $D_{\frac{1}{R}}\setminus\{0\}$ onto $\mathbb{C}\setminus\overline{D_R}$ and is bi-Lipschitz on compact subsets away from 0. Hence $W \in L^{2}_{loc}(D_{\frac{1}{R}}\setminus\{0\})$. Moreover, the standard chain rule for Sobolev functions (see [24, p. 52]) gives

$$v = u\circ F \in W^{1,p}_{loc}(D_{\frac{1}{R}}\setminus\{0\}),$$

and

$$\bar{\partial}_w v(w) = -\frac{1}{\bar{w}^2}(\bar{\partial}_z u)\left(\frac{1}{w}\right) = W(w)v(w) \ \text{ a.e. on } \ D_{\frac{1}{R}}\setminus\{0\}.$$

$\square$

Lemma 2.6. *Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be a domain, and write $z = (z_1, z') \in \mathbb{C}\times\mathbb{C}^{n-1}$. For $z' \in \mathbb{C}^{n-1}$, set $\Omega_{z'} := \{\zeta \in \mathbb{C} : (\zeta, z') \in \Omega\}$. Suppose that $u \in L^{p}_{loc}(\Omega)$, $f \in L^{q}_{loc}(\Omega)$ for some $p, q \geq 1$ and that*

$$\bar{\partial}_{z_1}u = f \quad on \quad \Omega$$

in the sense of distributions. Then, for almost every z', $u(\,\cdot\,,z') \in L^{p}_{loc}(\Omega_{z'})$, $f(\,\cdot\,,z') \in L^{q}_{loc}(\Omega_{z'})$ and

$$\bar{\partial}_{\zeta}u(\zeta,z') = f(\zeta,z') \quad on \quad \Omega_{z'} \tag{2.3}$$

in the sense of distributions. If, in addition, $u \in W^{1,p}_{loc}(\Omega)$, then $u(\,\cdot\,,z') \in W^{1,p}_{loc}(\Omega_{z'})$ for almost every z'.

Proof. It suffices to argue locally. Let $D \subset \mathbb{C}$ and $U \subset \mathbb{C}^{n-1}$ be relatively compact open sets such that $D \times U \Subset \Omega$. For $\varphi \in C_c^\infty(D)$ and $\psi \in C_c^\infty(U)$, testing the distributional identity $\bar{\partial}_{z_1} u = f$ against $\varphi(\zeta)\psi(z')$ gives

$$\int_U \psi(z') \left[\int_D f(\zeta, z')\varphi(\zeta)\, dv_\zeta + \int_D u(\zeta, z')\bar{\partial}_\zeta \varphi(\zeta)\, dv_\zeta \right] dv_{z'} = 0.$$

By Fubini's theorem, the expression in brackets vanishes for almost every $z' \in U$, for each fixed φ. A countable dense family of test functions in $C_c^\infty(D)$ allows the exceptional null set in z' to be chosen independently of φ. Hence, (2.3) holds on D in the sense of distributions for almost every $z' \in U$.

Moreover, Fubini's theorem gives $u(\,\cdot\,, z') \in L^p(D)$, $f(\,\cdot\,, z') \in L^q(D)$ for almost every $z' \in U$. Finally, if $u \in W^{1,p}_{loc}(\Omega)$, then the standard slicing property of Sobolev functions yields $u(\,\cdot\,, z') \in W^{1,p}_{loc}(\Omega_{z'})$. □

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1, which provides a quantitative lower bound for the sup-norm of solutions to (1.2) near infinity. We note that, by Lemma 2.3 part 2 and the boundedness of the potential V, the continuity assumption on u in Theorem 1.1 may be weakened to $u \in L^1_{loc}(\mathbb{C}^n)$. We retain the continuity assumption there and below only to ensure that the sup-norm of u appearing in the statement is well defined. As in [11] for the case $n = 1$, the main tools of the proof are the three-circle theorem for holomorphic functions and the scaling argument of Bourgain-Kenig [2].

We begin with a higher-dimensional version of Hadamard's three-circle theorem for holomorphic functions.

Lemma 3.1. *Let h be holomorphic in $B_2 \subset \mathbb{C}^n$. Then for any $0 < r < r_1 < r_2 < 2$,*

$$\|h\|_{L^\infty(B_{r_1})} \le \|h\|^{\theta}_{L^\infty(B_r)} \|h\|^{1-\theta}_{L^\infty(B_{r_2})},$$

where $\theta = \frac{\ln r_2 - \ln r_1}{\ln r_2 - \ln r}$.

Proof. For each $\zeta \in S^{2n-1}$, define $h^\zeta(w) := h(w\zeta), w \in D_2$. Then h^ζ is holomorphic in D_2. By Hadamard's three-circle theorem

$$\|h^\zeta\|_{L^\infty(D_{r_1})} \le \|h^\zeta\|^{\theta}_{L^\infty(D_r)} \|h^\zeta\|^{1-\theta}_{L^\infty(D_{r_2})}.$$

Taking the supremum over $\zeta \in S^{2n-1}$ gives

$$\sup_{\zeta \in S^{2n-1}} \|h^\zeta\|_{L^\infty(D_{r_1})} \le \sup_{\zeta \in S^{2n-1}} \|h^\zeta\|^{\theta}_{L^\infty(D_r)} \sup_{\zeta \in S^{2n-1}} \|h^\zeta\|^{1-\theta}_{L^\infty(D_{r_2})}.$$

The desired estimate then follows from the fact that, for every $0 < \rho < 2$,

$$\|h\|_{L^\infty(B_\rho)} = \sup_{\zeta \in S^{2n-1}} \|h^\zeta\|_{L^\infty(D_\rho)}.$$

□

We next apply the three-circle theorem above to derive a lower bound for the sup-norm of solutions to (1.2) on bounded domains. To this end, we first relate a solution of (1.2) to a holomorphic function. When $n = 1$, this is essentially immediate, since the $\bar\partial$-equation has no compatibility condition. In higher dimensions, the key observation is the $\bar\partial$-closedness of the potential V, relying on the following result of Gong-Rosay [7, Proposition A]. Combined with sup-norm estimates for the $\bar\partial$-equation on balls, this allows us to obtain an upper bound for the maximal vanishing order of solutions to (1.2) on balls.

Theorem 3.2. *[7] Let Ω be a domain in $\mathbb{C}^n$. Suppose $u \in C(\Omega)$ satisfies $\bar\partial u = Vu$ on Ω in the sense of distributions for some $(0,1)$-form $V \in L^\infty(\Omega)$. Then the zero set $u^{-1}(0)$ of u is a complex analytic variety.*

Lemma 3.3. *Let $u \in C(B_2)$ and V be a $(0,1)$-form on B_2 such that*

$$\|V\|_{L^\infty(B_2)} \le M$$

for a constant $M > 0$. Suppose that

$$\|u\|_{L^\infty(B_1)} \ge 1 \quad and \quad \|u\|_{L^\infty(B_2)} \le e^{C_0 M}. \tag{3.1}$$

for a constant $C_0 > 0$ and u satisfies

$$\bar\partial u = Vu \quad on \quad B_2$$

in the sense of distributions. Then there exist two positive constants C_1 and C_2 dependent only on C_0 and n, such that for all $0 < r < 1$,

$$\|u\|_{L^\infty(B_r)} \ge C_1^M r^{C_2 M}.$$

Proof. Let $S := \{z \in B_2 : u(z) = 0\} \subsetneq B_2$. Since u is continuous, $B_2 \setminus S$ is open. For every $z_0 \in B_2 \setminus S$, we may choose a sufficiently small neighborhood U of z_0 such that u does not vanish on U and a branch of $\log u$ is defined there. By Lemma 2.3, we have $u \in W^{1,p}_{loc}(U)$ for all $p < \infty$. The Sobolev chain rule [24, p. 48] then gives $\log u \in W^{1,p}_{loc}(U)$ and

$$\bar\partial(\log u) = \frac{\bar\partial u}{u} = V \quad \text{on} \quad U.$$

Hence $\bar\partial V = 0$ on U in the sense of distributions. Therefore, V is $\bar\partial$-closed on $B_2 \setminus S$. Since $V \in L^\infty(B_2)$, Theorem 3.2 of Gong and Rosay implies that S is a complex analytic variety in B_2. It then follows from Demailly's removable singularity result [5, Lemma 6.9] that

$$\bar\partial V = 0 \quad \text{on} \quad B_2$$

in the sense of distributions. Consequently, the well-known sup-norm estimates for the $\bar\partial$-equation on balls provide a function $\lambda \in L^\infty(B_2)$ such that

$$\bar\partial \lambda = V \quad \text{on} \quad B_2$$

with

$$\|\lambda\|_{L^\infty(B_2)} \le A(n)\|V\|_{L^\infty(B_2)} = A(n)M \tag{3.2}$$

where $A(n) > 0$ depends only on n; see, for instance, [13, pp. 94-95]. Moreover, the ellipticity of $\bar{\partial}$ implies that $\lambda \in W^{1,p}_{loc}(B_2)$ for all $p < \infty$.

Since $u \in W^{1,p}_{loc}(B_2)$ for all $p < \infty$ by Lemma 2.3, we can apply Lemma 2.4 to obtain

$$\bar{\partial}(ue^{-\lambda}) = \bar{\partial}ue^{-\lambda} - ue^{-\lambda}\bar{\partial}\lambda = Vue^{-\lambda} - Vue^{-\lambda} = 0 \quad \text{on} \quad B_2.$$

Thus

$$h := ue^{-\lambda}$$

is holomorphic on B_2. By (3.1) and (3.2), we have

$$\begin{aligned} \|h\|_{L^\infty(B_1)} &\geq e^{-A(n)M}\|u\|_{L^\infty(B_1)} \geq e^{-A(n)M} \geq e^{-\tilde{C}M}; \\ \|h\|_{L^\infty(B_2)} &\leq e^{A(n)M}\|u\|_{L^\infty(B_2)} \leq e^{(A(n)+C_0)M} \leq e^{\tilde{C}M}, \end{aligned} \tag{3.3}$$

where $\tilde{C} := A(n) + C_0$.

Applying Lemma 3.1 with $r_1 = 1$ and $r_2 = \frac{3}{2}$, we obtain from (3.3) that for all $0 < r < 1$,

$$e^{-\tilde{C}M} \leq \|h\|_{L^\infty(B_1)} \leq \|h\|^{\theta}_{L^\infty(B_r)}\|h\|^{1-\theta}_{L^\infty(B_{3/2})} \leq \|h\|^{\theta}_{L^\infty(B_r)}e^{(1-\theta)\tilde{C}M},$$

where $\theta = \frac{\ln 3 - \ln 2}{\ln 3 - \ln 2 - \ln r}$. Hence

$$\|h\|_{L^\infty(B_r)} \geq e^{-\frac{2\tilde{C}M}{\theta}}e^{\tilde{C}M}.$$

Using (3.2), $u = he^{\lambda}$ and $\tilde{C} > A(n)$, we obtain

$$\|u\|_{L^\infty(B_r)} \geq e^{-A(n)M}\|h\|_{L^\infty(B_r)} \geq e^{-\frac{2\tilde{C}M}{\theta}} = C_1^M r^{C_2 M},$$

where $C_1 = e^{-2\tilde{C}}$ and $C_2 = \frac{2\tilde{C}}{\ln 3 - \ln 2}$. Both constants depend only on C_0 and n. □

It is worth noting that the continuity of u is essential in order to conclude the $\bar{\partial}$-closedness of the $(0,1)$-form $\frac{\bar{\partial}u}{u}$ in the proof of Lemma 3.3. Indeed, the following example shows that, without continuity, this form may be $\bar{\partial}$-closed off a closed set but fail to remain $\bar{\partial}$-closed across it.

Example 1. *Let*

$$F = \{z \in \mathbb{C}^2 : \operatorname{Re} z_1 = 0\}.$$

Then F is a closed real hypersurface in $\mathbb{C}^2$ of real codimension 1. Define

$$u_0(z) = \begin{cases} 1, & \operatorname{Re} z_1 < 0, \\ e^{\bar{z}_2}, & \operatorname{Re} z_1 > 0. \end{cases}$$

Then u_0 is discontinuous across F and smooth on $\mathbb{C}^2 \setminus F$. On $\mathbb{C}^2 \setminus F$, we have

$$\frac{\bar{\partial}u_0}{u_0} = H(\operatorname{Re} z_1)\, d\bar{z}_2,$$

where H denotes the Heaviside function. Thus $\frac{\bar{\partial}u_0}{u_0}$ is a bounded $(0,1)$-form and is $\bar{\partial}$-closed off F. However, in the sense of distributions,

$$\bar{\partial}\left(\frac{\bar{\partial}u_0}{u_0}\right) = \frac{1}{2}\,\delta_{\{\operatorname{Re} z_1 = 0\}}\, d\bar{z}_1 \wedge d\bar{z}_2 \neq 0.$$

Therefore $\frac{\bar{\partial}u_0}{u_0}$ is not $\bar{\partial}$-closed across F.

Finally, the scaling method of Bourgain-Kenig [2] converts the preceding lower bound for solutions of (1.2) on bounded domains into a corresponding lower bound near infinity as stated in Theorem 1.1.

Proof of Theorem 1.1: We first prove the theorem under the normalization $\|V\|_{L^\infty(\mathbb{C}^n)} \le 1$. Fix $z_0 \in \mathbb{C}^n$ with $|z_0| = R \gg 1$, and define $u_R(z) := u(Rz + z_0)$ and $V_R = R \cdot V(Rz + z_0)$ on B_2. Then for all sufficiently large R,

$$\begin{aligned}&\|u_R\|_{L^\infty(B_2)} \le e^{C_0(2R+|z_0|)} \le e^{3C_0R};\\ &\|V_R\|_{L^\infty(B_2)} \le R,\end{aligned}$$

and

$$\bar\partial u_R(z) = R \cdot V(Rz + z_0)u_R(z) = V_R(z)u_R(z), \quad z \in B_2.$$

Since $u_R(-\frac{z_0}{R}) = u(0) = 1$,

$$\|u_R\|_{L^\infty(B_1)} \ge u_R\left(-\frac{z_0}{R}\right) = 1.$$

Applying Lemma 3.3, with $M = R$ and C_0 there replaced by $3C_0$, we obtain constants $C_1, C_2 > 0$, depending only on C_0 and n, such that for $0 < r < 1$,

$$\|u_R\|_{L^\infty(B_r)} \ge C_1^R r^{C_2R}.$$

Let $R_0 > 1$ be such that $C_1 \ge R_0^{-C_2}$. Furthermore, for every $z_0 \in \mathbb{C}^n$ with $|z_0| = R \ge R_0$, taking $r = R^{-1}$ gives

$$\sup_{|z-z_0|<1} |u| = \|u_R\|_{L^\infty(B_{R^{-1}})} \ge C_1^R R^{-C_2R} \ge R^{-2C_2R} = e^{-2C_2R\ln R}.$$

Thus the desired lower bound follows in the case $\|V\|_{L^\infty(\mathbb{C}^n)} \le 1$, with $C = 2C_2$ depending only on C_0 and n.

It remains to remove the normalization on V. Let $M := \|V\|_{L^\infty(\mathbb{C}^n)} > 1$ and define $v(z) := u\left(\frac{z}{M}\right), z \in \mathbb{C}^n$. Then $v(0) = u(0) = 1$,

$$|v(z)| = \left|u\left(\frac{z}{M}\right)\right| \le e^{\frac{C_0}{M}|z|} \le e^{C_0|z|}, \qquad |z| \gg 1,$$

and

$$\bar\partial v(z) = \frac{1}{M}V\left(\frac{z}{M}\right)u\left(\frac{z}{M}\right) = \tilde V(z)v(z) \qquad \text{on } \mathbb{C}^n,$$

with $\tilde V := \frac{1}{M}V\left(\frac{\cdot}{M}\right)$. In particular, $\|\tilde V\|_{L^\infty(\mathbb{C}^n)} = 1$. Applying the normalized case to v, we obtain

$$\inf_{|z_0|=R} \sup_{|z-z_0|<1} |v(z)| \ge e^{-CR\ln R}, \quad R \gg 1,$$

where C depends only on C_0 and n. Consequently, since $M > 1$,

$$\begin{aligned}\inf_{|z_0|=R} \sup_{|z-z_0|<1} |u(z)| &= \inf_{|z_0|=R} \sup_{|z-Mz_0|<M} |v(z)| \ge \inf_{|z_0|=R} \sup_{|z-Mz_0|<1} |v(z)|\\ &= \inf_{|z_0|=MR} \sup_{|z-z_0|<1} |v(z)| \ge e^{-CMR\ln(MR)} \ge e^{-\tilde CR\ln R}, \quad R \gg 1,\end{aligned}$$

for some $\tilde C > 0$ depending only on C_0, n and $\|V\|_{L^\infty(\mathbb{C}^n)}$. The proof is complete. $\square$

Several remarks concerning the statement of Theorem 1.1 are in order.

Remark 3.4. *The $R \ln R$ dependence in the exponent of the lower bound in Theorem 1.1 is optimal up to a multiplicative constant, even in the holomorphic case with $n = 1$ and $V \equiv 0$. This is demonstrated by the following family of entire functions.*

Fix $0 < c < C_0$. For each sufficiently large $R > 1$, choose $z_R \in \mathbb{C}$ with $|z_R| = R$, and let

$$m = \lfloor cR \rfloor. \tag{3.4}$$

Define

$$u_R(z) = \left(\frac{z_R - z}{z_R}\right)^m = \left(1 - \frac{z}{z_R}\right)^m, \qquad z \in \mathbb{C}.$$

The family $\{u_R\}$ is entire, $u_R(0) = 1$ and

$$\bar{\partial} u_R = 0 = V u_R \qquad \text{on } \mathbb{C}$$

with $V \equiv 0$. Moreover, by (3.4)

$$|u_R(z)| \leq \left(1 + \frac{|z|}{R}\right)^m \leq e^{\frac{m}{R}|z|} \leq e^{c|z|} \leq e^{C_0|z|} \qquad \text{on } \mathbb{C}.$$

On the other hand,

$$\sup_{|z - z_R| < 1} |u_R(z)| = \sup_{|z - z_R| < 1} \left|\frac{z - z_R}{z_R}\right|^m = R^{-m}.$$

Consequently,

$$\inf_{|\zeta| = R} \sup_{|z - \zeta| < 1} |u_R(z)| \leq \sup_{|z - z_R| < 1} |u_R(z)| = R^{-m} = e^{-m \ln R} \leq e^{-\frac{c}{2} R \ln R}.$$

Here, the last inequality follows from (3.4), and hence $m \geq \frac{c}{2} R$ for all sufficiently large R. Thus, the family $\{u_R\}_{R \gg 1}$ satisfies the assumptions of Theorem 1.1 uniformly in R and exhibits decay of order $e^{-cR \ln R}$. Consequently, the $R \ln R$ exponent in Theorem 1.1 cannot, in general, be improved even for entire holomorphic functions.

Remark 3.5. *The conclusion of Theorem 1.1 generally fails if the domain $\mathbb{C}^n$ is replaced by the complement of a compact set. This is already evident in one complex dimension. Let $\Omega = \mathbb{C} \setminus \overline{D_1}$. For each $m \in \mathbb{N}$, define*

$$u_m(z) := \left(\frac{2}{z}\right)^m, \qquad z \in \Omega.$$

Then u_m is holomorphic on Ω, satisfies $u_m(2) = 1$, and

$$\bar{\partial} u_m = 0 = V u_m \qquad \text{on } \Omega$$

with $V \equiv 0$. Moreover,

$$|u_m(z)| \leq 1 \leq e^{C_0|z|}, \qquad |z| \geq 2,$$

for any $C_0 > 0$.

On the other hand, for every $R > 2$, and $z_0 \in \mathbb{C}$ with $|z_0| = R$,

$$\sup_{|z-z_0|<1} |u_m(z)| = \left(\frac{2}{R-1}\right)^m.$$

Thus, given any $C > 0$, by choosing $m = \lceil 2CR \rceil$ and then taking R sufficiently large, we obtain

$$\sup_{|z-z_0|<1} |u_m(z)| \le \left(\frac{2}{R-1}\right)^{2CR} = e^{-2CR\ln\frac{R-1}{2}} < e^{-CR\ln R}.$$

Hence no lower bound of the form appearing in Theorem 1.1, with a constant depending only on C_0 and $\|V\|_{L^\infty}$, can hold on the complement of a compact set.

Remark 3.6. *The growth assumption $|u(z)| \le e^{C_0|z|}$ for sufficiently large $|z|$ cannot in general be omitted. Indeed, consider*

$$u(z) = e^{e^{z_1}-1}, \qquad z = (z_1, \ldots, z_n) \in \mathbb{C}^n.$$

Then u is entire, $u(0) = 1$, and $\bar\partial u = 0$. The function u does not satisfy the above exponential growth condition, since $u(R, 0, \ldots, 0) = e^{e^R-1}$.

Moreover, for $R > \pi$, set

$$x_R = \sqrt{R^2 - \pi^2}, \qquad z_R = (x_R + i\pi, 0, \ldots, 0),$$

so that $|z_R| = R$. If $|z - z_R| < 1$, write $z_1 = x_R + s + i(\pi + t)$ where $|s| < 1$ and $|t| < 1$. Then

$$\operatorname{Re}(e^{z_1}) = -e^{x_R+s}\cos t \le -e^{x_R-1}\cos 1.$$

It follows that

$$\sup_{|z-z_R|<1} |u(z)| \le e^{-e^{x_R-1}\cos 1 - 1}.$$

Since $x_R \sim R$ as $R \to \infty$, the right-hand side decays faster than $e^{-CR\ln R}$ for every $C > 0$. Consequently, $\inf_{|z_0|=R} \sup_{|z-z_0|<1} |u(z)|$ does not admit a lower bound of the form $e^{-CR\ln R}$ with a fixed constant C.

A similar example with a nontrivial bounded potential is obtained by setting

$$u(z) = f(z)e^{g(z)}.$$

with $f = e^{e^{z_1}-1}$, and $g(z) = \sin(\operatorname{Re} z_1)$. Then $u \in C^\infty(\mathbb{C}^n)$, $u(0) = 1$, and

$$\bar\partial u = (\bar\partial g)u = Vu,$$

where $V = \frac{1}{2}\cos(\operatorname{Re} z_1)\, d\bar z_1$. In particular,

$$\|V\|_{L^\infty(\mathbb{C}^n)} \le \frac{1}{2}.$$

Since $-1 \le g \le 1$,

$$e^{-1}|f(z)| \le |u(z)| \le e|f(z)|.$$

Thus u fails the exponential growth condition of Theorem 1.1, just as f does. Moreover,

$$\sup_{|z-z_R|<1} |u(z)| \le e \sup_{|z-z_R|<1} |f(z)| \le e^{-e^{x_R-1}\cos 1},$$

which again decays faster than $e^{-CR\ln R}$ for every $C > 0$.

One may also consider analogous bounds for the full gradient operator. As shown below, the full-gradient setting yields a substantially stronger lower bound than the $\bar{\partial}$ case.

Lemma 3.7. *Suppose that $u = (u_1, \dots, u_N) \in W^{1,1}_{loc}(\mathbb{R}^n)$ satisfies*

$$|\nabla u(x)| \le C|u(x)| \qquad a.e.\ on\ \ \mathbb{R}^n. \tag{3.5}$$

Then u admits a locally Lipschitz representative, still denoted by u, such that either $u \equiv 0$, or else u is nowhere vanishing and

$$|u(0)|e^{-C|x|} \le |u(x)| \le |u(0)|e^{C|x|}, \qquad x \in \mathbb{R}^n.$$

Proof. By the Sobolev embedding theorem and (3.5), a standard bootstrap argument yields $u \in W^{1,\infty}_{loc}(\mathbb{R}^n)$. Thus u admits a locally Lipschitz representative. Set

$$w(x) = |u(x)|.$$

Then $w \in W^{1,\infty}_{loc}(\mathbb{R}^n)$ and

$$|\nabla w| \le |\nabla u| \le Cw \qquad \text{a.e. in } \ \mathbb{R}^n.$$

For $\varepsilon > 0$, define

$$v_\varepsilon(x) = \log\big(w(x) + \varepsilon\big).$$

Then

$$|\nabla v_\varepsilon| = \frac{|\nabla w|}{w + \varepsilon} \le \frac{Cw}{w + \varepsilon} \le C \qquad \text{a.e. in } \ \mathbb{R}^n.$$

Hence v_ε satisfies

$$|v_\varepsilon(x) - v_\varepsilon(y)| \le C|x - y|, \qquad x, y \in \mathbb{R}^n.$$

Equivalently,

$$e^{-C|x-y|} \le \frac{|u(x)| + \varepsilon}{|u(y)| + \varepsilon} \le e^{C|x-y|}. \tag{3.6}$$

Suppose that $u(y) = 0$ for some $y \in \mathbb{R}^n$. Then

$$|u(x)| + \varepsilon \le e^{C|x-y|}\varepsilon.$$

Letting $\varepsilon \to 0$ gives $u(x) = 0$ for every $x \in \mathbb{R}^n$. Thus either $u \equiv 0$, or u is nowhere vanishing. In the latter case, letting $\varepsilon \to 0$ in (3.6) yields

$$e^{-C|x-y|} \le \frac{|u(x)|}{|u(y)|} \le e^{C|x-y|}, \qquad x, y \in \mathbb{R}^n.$$

Taking $y = 0$, we obtain the desired inequality, which completes the proof. $\square$

As an application, Theorem 1.1 immediately yields the following unique continuation property for (1.2) at infinity. The $|z|\ln|z|$ decay assumption here is not optimal for this purpose: as Theorem 1.2 will show, a sufficiently rapid linear exponential decay already forces triviality.

Corollary 3.8. *Suppose $u \in L^1_{loc}(\mathbb{C}^n)$ satisfies*

$$\bar{\partial} u = V u \quad on \quad \mathbb{C}^n$$

in the sense of distributions for some $(0,1)$-form $V \in L^\infty(\mathbb{C}^n)$. Then there exists a constant $\tilde{C} > 0$ such that whenever

$$|u(z)| \le e^{-\tilde{C}|z||\ln |z||}, \qquad |z| \gg 1,$$

then $u \equiv 0$. In particular, if

$$|u(z)| \le e^{-|z|^{1+\epsilon}}, \quad |z| \gg 1$$

for some $\epsilon > 0$, then $u \equiv 0$.

Proof. By Lemma 2.3, $u \in C(\mathbb{C}^n)$. We argue by contradiction that $u \not\equiv 0$. Choose $a \in \mathbb{C}^n$ such that $u(a) \neq 0$ and define

$$v(z) := \frac{u(z+a)}{u(a)}, \qquad \tilde{V}(z) := V(z+a).$$

Then

$$v(0) = 1, \qquad \|\tilde{V}\|_{L^\infty(\mathbb{C}^n)} = \|V\|_{L^\infty(\mathbb{C}^n)},$$

and

$$\bar{\partial} v = \tilde{V} v \qquad \text{on} \quad \mathbb{C}^n$$

in the sense of distributions. Moreover, since v tends to zero at infinity, it satisfies, for example,

$$|v(z)| \le e^{|z|} \qquad \text{for } |z| \gg 1.$$

Theorem 1.1 then provides a constant $C > 0$ dependent on $\|V\|_{L^\infty(\mathbb{C}^n)}$ and n such that, for all sufficiently large R, and every $z_0 \in \mathbb{C}^n$ with $|z_0| = R$.

$$\sup_{|z-z_0|<1} |v(z)| \ge e^{-CR\ln R}.$$

On the other hand, for $|z_0| = R$ and $|z - z_0| < 1$, $|z + a| \ge R - |a| - 1$. Hence the assumed decay of u gives

$$\sup_{|z-z_0|<1} |v(z)| \le \frac{1}{|u(a)|} e^{-\tilde{C}(R-|a|-1)\ln(R-|a|-1)}$$

for all sufficiently large R. Since $(R-|a|-1)\ln(R-|a|-1) \sim R\ln R$ as $R \to \infty$, this contradicts the preceding lower bound provided $\tilde{C} > C$. Therefore $u \equiv 0$. Finally, the second assertion follows from the first since for every $\epsilon > 0$ and every $\tilde{C} > 0$, $|z|^{1+\epsilon} \ge \tilde{C}|z|\ln|z|$ for all sufficiently large $|z|$. $\square$

4 Proofs of Theorems 1.2 and 1.4

Instead of proving Theorem 1.2 directly, we establish unique continuation for a large class of radially decaying potentials described below, from which Theorem 1.2 follows immediately.

Theorem 4.1. *Let $\phi : [R_0, \infty) \to (0, \infty)$ be a nonincreasing function for some $R_0 \geq 0$, and set*

$$\Phi(r) := \int_{R_0}^{r} \phi(s) ds.$$

Assume that

$$\frac{\Phi(r)}{\ln r} \to \infty \qquad \text{as } r \to \infty. \tag{4.1}$$

Suppose $u \in L^1_{loc}(\mathbb{C}^n \setminus \overline{B_R})$ satisfies

$$\bar{\partial} u = V u \qquad \text{on } \mathbb{C}^n \setminus \overline{B_R}$$

in the sense of distributions, where $V \in L^\infty(\mathbb{C}^n \setminus \overline{B_R})$ is a $(0,1)$-form satisfying

$$|V(z)| \leq C_0 \phi(|z|), \qquad |z| \gg 1,$$

for some $C_0 > 0$. If

$$|u(z)| \leq e^{-C\Phi(|z|)}, \qquad |z| \gg 1,$$

for some $C > 2C_0$, then u vanishes identically.

The condition (4.1) is satisfied by many natural decay rates, as illustrated by the following examples, which yield corresponding unique continuation results.

Example 2. *Suppose $u \in L^1_{loc}(\mathbb{C}^n \setminus \overline{B_R})$ satisfies $\bar{\partial} u = Vu$ on $\mathbb{C}^n \setminus \overline{B_R}$ in the sense of distributions for some $(0,1)$-form $V \in L^\infty(\mathbb{C}^n \setminus \overline{B_R})$.*

1. *Let*

   $$\phi(r) \equiv 1 \qquad \text{and} \qquad C_0 = \|V\|_{L^\infty(\mathbb{C}^n \setminus \overline{B_R})}.$$

   Then

   $$\Phi(r) = \int_{R_0}^{r} \phi(s)\, ds = r + O(1),$$

   and the hypothesis on V in Theorem 4.1 is automatically satisfied. Hence, if

   $$|u(z)| \leq e^{-C|z|}, \qquad |z| \gg 1$$

   for some $C > 2\|V\|_{L^\infty(\mathbb{C}^n \setminus \overline{B_R})}$, then $u \equiv 0$. Thus Theorem 4.1 recovers Theorem 1.2.

2. *Let*

   $$\phi(r) = \frac{1}{(\ln r)^\alpha}, \qquad \alpha > 0.$$

   Then for $R_0 > 1$,

   $$\Phi(r) = \int_{R_0}^{r} \phi(s)\, ds = \frac{r}{(\ln r)^\alpha} \left(1 + O\left(\frac{1}{\ln r}\right)\right).$$

   Consequently, Theorem 4.1 implies that if

   $$|V(z)| \leq \frac{C_0}{(\ln |z|)^\alpha}, \qquad |z| \gg 1,$$

   and

   $$|u(z)| \leq e^{-\frac{C|z|}{(\ln |z|)^\alpha}}, \qquad |z| \gg 1,$$

   for some $C > 2C_0$, then $u \equiv 0$.

3. *Let*
$$\phi(r) = \frac{(\ln r)^\beta}{r}, \qquad \beta > 0.$$
Then
$$\Phi(r) = \int_{R_0}^r \phi(s)\,ds = \frac{(\ln r)^{\beta+1}}{\beta+1} + O(1).$$
Choose R_0 sufficiently large so that ϕ is nonincreasing when $r \geq R_0$. Hence, if
$$|V(z)| \leq C_0 \frac{(\ln|z|)^\beta}{|z|}, \qquad |z| \gg 1,$$
and
$$|u(z)| \leq e^{-\frac{C}{\beta+1}(\ln|z|)^{\beta+1}}, \qquad |z| \gg 1,$$
for some $C > 2C_0$, then $u \equiv 0$.

To prove Theorem 4.1, we first establish the result in one complex dimension and then use slicing to reduce the general case to almost every complex line parallel to a coordinate axis. In one dimension, the similarity principle below shows that the zeros of a nontrivial solution are isolated and have positive multiplicities, which implies a monotonicity property for the winding number on concentric circles. Combined with the equation in polar coordinates, this yields a lower bound on the radial mean of $\ln|u|$ that is incompatible with the decay assumption in Theorem 4.1.

Lemma 4.2. *Let $\Omega \subset \mathbb{C}$ be a domain. Suppose $u \in L^1_{loc}(\Omega)$ satisfies*
$$\bar\partial u = Vu \qquad \text{on} \quad \Omega$$
in the sense of distributions, where $V \in L^\infty_{loc}(\Omega)$. Then the following hold.

1. *For every relatively compact smooth domain $D \Subset \Omega$, there exist $\lambda \in W^{1,p}(D)$ for every $p < \infty$, and a holomorphic function h on D such that*
$$u = e^\lambda h \qquad \text{on} \quad D.$$
In particular, either $u \equiv 0$, or the zeros of u are isolated. If z_0 is a zero of u, its multiplicity is defined by
$$\operatorname{ord}_{z_0} u := \operatorname{ord}_{z_0} h, \tag{4.2}$$
and is a positive integer.

2. *Let $0 < r_1 < r_2$ be such that $\{z : r_1 \leq |z| \leq r_2\} \Subset \Omega$ and u does not vanish on the circles $|z| = r_1$ and $|z| = r_2$. Denote by $k(r_j)$ the winding number of the curve $\theta \mapsto u(r_j e^{i\theta})$ about the origin. Then*
$$k(r_2) - k(r_1) = \sum_{r_1 < |z| < r_2} \operatorname{ord}_z u \geq 0.$$

Proof. By Lemma 2.3 part 2, $u \in W^{1,p}_{loc}(\Omega)$ for every $p < \infty$, and in particular u is continuous. Fix a relatively compact smooth domain $D \Subset \Omega$. Let
$$\lambda(z) = -\frac{1}{\pi}\int_D \frac{V(\zeta)}{\zeta - z}\,dv_\zeta.$$

Then $\lambda \in W^{1,p}(D)$ for every $p < \infty$, and solves $\bar{\partial}\lambda = V$ on D. See, for instance, [23] and [19]. In particular, $\lambda \in C(D)$. By Lemma 2.4,

$$h := e^{-\lambda} u$$

is holomorphic on D. Since e^{λ} is continuous and nowhere vanishing, u and h have the same zeros. Thus, unless $u \equiv 0$, its zeros are isolated.

For a zero z_0 of u, the definition (4.2) is independent of the factorization. Indeed, if $u = e^{\lambda_1} h_1 = e^{\lambda_2} h_2$ locally, then $\bar{\partial}(\lambda_1 - \lambda_2) = 0$, so that $\lambda_1 - \lambda_2$ is holomorphic, and

$$h_2 = e^{\lambda_1 - \lambda_2} h_1.$$

Since the holomorphic factor $e^{\lambda_1 - \lambda_2}$ is nowhere vanishing, h_1 and h_2 have the same order at z_0. Consequently, every zero of u has positive integer multiplicity.

For the second assertion, choose a relatively compact smooth domain $U \Subset \Omega$ containing $\{z : r_1 \le |z| \le r_2\}$. By the preceding construction, $u = e^{\lambda} h$ on U, with h holomorphic. Since e^{λ} has winding number zero on every closed curve, u and h have the same winding number on the two boundary circles. Hence the argument principle gives

$$k(r_2) - k(r_1) = \sum_{r_1 < |z| < r_2} \operatorname{ord}_z h = \sum_{r_1 < |z| < r_2} \operatorname{ord}_z u \ge 0.$$

This proves the result. $\square$

Proof of Theorem 4.1: We first consider the case $n = 1$. By Lemma 2.3 part 2, $u \in W^{1,p}_{loc}(\mathbb{C} \setminus \overline{D_R})$ for every $p < \infty$, and in particular u is continuous.

Suppose that $u \not\equiv 0$. By Lemma 4.2, the zeros of u are isolated. Choose $r_0 > R$ sufficiently large so that the assumed estimate for V holds for $|z| > r_0$ and u has no zeros on $|z| = r_0$. For $r > r_0$ such that u has no zeros on $|z| = r$, let $k(r)$ denote the winding number of $\theta \mapsto u(re^{i\theta})$ about the origin, and define

$$m(r) := \frac{1}{2\pi} \int_0^{2\pi} \ln |u(re^{i\theta})| \, d\theta.$$

By the local factorization in the proof of Lemma 4.2, $\ln |u| = \ln |h| + \operatorname{Re} \lambda$, where $h \not\equiv 0$ is holomorphic. Since $\ln |h| \in W^{1,1}_{loc}$, it follows that $\ln |u| \in W^{1,1}_{loc}$, and hence m is locally absolutely continuous. Moreover, for almost every $r > r_0$, the winding number

$$k(r) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\partial_\theta u(re^{i\theta})}{u(re^{i\theta})} \, d\theta.$$

Since $\bar{\partial} = \frac{e^{i\theta}}{2} \left(\partial_r + \frac{i}{r} \partial_\theta \right)$ in polar coordinates, for almost every $r > r_0$, the equation (1.2) is rewritten as

$$\frac{\partial_r u}{u} + \frac{i}{r} \frac{\partial_\theta u}{u} = 2e^{-i\theta} V.$$

Averaging over θ and taking real parts, we obtain

$$m'(r) - \frac{k(r)}{r} = \frac{1}{2\pi} \int_0^{2\pi} 2 \operatorname{Re} \left(e^{-i\theta} V(re^{i\theta}) \right) \, d\theta.$$

Therefore, by the monotonicity of k in Lemma 4.2,

$$m'(r) \ge \frac{k(r)}{r} - 2C_0 \phi(r) \ge \frac{k(r_0)}{r} - 2C_0 \phi(r).$$

Integrating from r_0 to r yields

$$m(r) \geq m(r_0) + k(r_0) \ln \frac{r}{r_0} - 2C_0 \int_{r_0}^{r} \phi(s)ds.$$

Making use of the definition of Φ and its assumption,

$$m(r) \geq -2C_0\Phi(r) + O(\ln r) \geq -(2C_0 + o(1))\Phi(r) \qquad \text{as } r \to \infty.$$

On the other hand, the assumed decay of u gives

$$m(r) \leq -C\Phi(r)$$

for all sufficiently large r. Since $C > 2C_0$, these two estimates are incompatible as $r \to \infty$. Hence $u \equiv 0$ on $\mathbb{C} \setminus \overline{D_R}$.

We now assume $n \geq 2$. Writing $V = \sum_{j=1}^{n} V_j d\bar{z}_j$, the equation (1.2) is equivalent to

$$\bar{\partial}_{z_j} u = V_j u, \qquad j = 1, \dots, n \tag{4.3}$$

in the sense of distributions on $\mathbb{C}^n \setminus \overline{B_R}$. We consider slices of $\mathbb{C}^n$ parallel to the z_1-axis and $z = (z_1, z')$, with $z' \in \mathbb{C}^{n-1}$. For almost every fixed $z' \in \mathbb{C}^{n-1}$ with $|z'| < \frac{R}{2}$, Lemma 2.6, applied to the first equation of (4.3), gives

$$\tilde{u}(\zeta) := u(\zeta, z') \in L^1_{loc}(\mathbb{C} \setminus \overline{D_{\sqrt{R^2 - |z'|^2}}}), \qquad \tilde{V}(\zeta) := V_1(\zeta, z') \in L^\infty(\mathbb{C} \setminus \overline{D_{\sqrt{R^2 - |z'|^2}}})$$

and

$$\bar{\partial}_\zeta \tilde{u} = \tilde{V}\tilde{u} \qquad \text{on } \ \mathbb{C} \setminus \overline{D_{\sqrt{R^2 - |z'|^2}}}$$

in the sense of distributions.

Let $\rho = \sqrt{|\zeta|^2 + |z'|^2}$. Since $\rho \geq |\zeta|$ and ϕ is nonincreasing, for sufficiently large $|\zeta|$,

$$|\tilde{V}(\zeta)| \leq C_0\phi(\rho) \leq C_0\phi(|\zeta|).$$

Moreover, since Φ is increasing,

$$|\tilde{u}(\zeta)| \leq e^{-C\Phi(\rho)} \leq e^{-C\Phi(|\zeta|)}.$$

The one-dimensional result therefore applies and yields $\tilde{u} \equiv 0$ for almost every $z' \in \mathbb{C}^{n-1}$ with $|z'| < \frac{R}{2}$. Consequently, by Fubini's theorem, $u = 0$ almost everywhere on the nonempty open set $\{|z'| < \frac{R}{2},\ |z_1| > \sqrt{R^2 - |z'|^2}\}$. Hence $u \equiv 0$ in $\mathbb{C}^n \setminus \overline{B_R}$ by the weak unique continuation property in Theorem 2.1 part 1. $\square$

Remark 4.3. *The condition (4.1) is essential. Indeed, in the borderline case*

$$\phi(r) = \frac{1}{r},$$

one has

$$\Phi(r) = \int_1^r \phi(s)\, ds = \ln r$$

with $R_0 = 1$, so that the hypothesis (4.1) fails. Moreover, for any $C > 0$, choose an integer $m \geq C$ and consider the nontrivial holomorphic function

$$u(z) = z^{-m}, \qquad |z| > 1.$$

Then

$$|u(z)| = |z|^{-m} \leq |z|^{-C} = e^{-C\Phi(|z|)}, \qquad |z| > 1.$$

Thus, when (4.1) is dropped, even an arbitrarily large constant C in the prescribed decay does not force vanishing. This shows that the borderline scale $\phi(r) = \frac{1}{r}$ cannot in general be included.

Remark 4.4. *The constant $2C_0$ in Theorem 4.1 is sharp. Indeed, let*

$$u(z) := e^{-2C_0\Phi(|z|)}, \qquad |z| > R.$$

Since Φ is locally absolutely continuous and $\Phi'(r) = \phi(r)$, for almost every $r > R \geq R_0$, we have $\bar{\partial}u = Vu$ almost everywhere on $|z| > R$, where $V(z) := -C_0\phi(|z|)\sum_{j=1}^{n} \frac{z_j}{|z|}\, d\bar{z}_j$. Thus

$$|V(z)| = C_0\phi(|z|).$$

Hence the strict threshold $C > 2C_0$ cannot be improved.

We next prove Theorem 1.4 for compactly supported potentials. The proof follows the approach of [4], while extending the result to higher dimensions and relaxing the assumption $V \in L^\infty(\mathbb{C})$ imposed there to $V \in L^p(\mathbb{C}^n)$ for some $p > 2$.

Proof of Theorem 1.4: We first prove the Theorem when $n = 1$. By Lemma 2.3 part 1, $u \in W^{1,p}_{loc}(\mathbb{C})$. In particular, u is continuous on $\mathbb{C}$. Since $\lim_{z\to\infty} u = 0$, it follows that u is bounded on $\mathbb{C}$.

Define

$$\lambda(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{V(\zeta)}{\zeta - z}\, dv_\zeta, \qquad z \in \mathbb{C}. \tag{4.4}$$

Since $V \in L^p(\mathbb{C})$ for some $p > 2$ and has compact support, we have $\lambda \in W^{1,p}_{loc}(\mathbb{C})$ and

$$\bar{\partial}\lambda = V \qquad \text{in } \ \mathbb{C}$$

in the sense of distributions. In particular, by the Sobolev embedding theorem, λ is continuous on $\mathbb{C}$. Moreover, the compact support of V implies from (4.4) that $\lim_{|z|\to\infty} \lambda(z) = 0$. Hence λ is bounded on $\mathbb{C}$.

By Lemma 2.4,

$$\bar{\partial}(ue^{-\lambda}) = 0 \qquad \text{in } \mathbb{C}$$

in the sense of distributions. Thus $ue^{-\lambda}$ is holomorphic on $\mathbb{C}$. On the other hand, since λ is bounded and $\lim_{|z|\to\infty} u(z) = 0$, we have

$$\lim_{z\to\infty} u(z)e^{-\lambda(z)} = 0.$$

Liouville's theorem therefore yields $ue^{-\lambda} \equiv 0$. Since $e^{-\lambda}$ never vanishes, it follows that $u \equiv 0$ on $\mathbb{C}$.

The case $n \geq 2$ follows from the one-dimensional result by the same coordinate slicing argument as in the proof of Theorem 4.1. Write $V = \sum_{j=1}^{n} V_j \, d\bar{z}_j$ and decompose $z = (z_1, z')$, where $z' \in \mathbb{C}^{n-1}$. For almost every fixed $z' \in \mathbb{C}^{n-1}$, Lemma 2.6 gives

$$\tilde{u}(\zeta) = u(\zeta, z') \in L^2_{loc}(\mathbb{C}), \qquad \tilde{V}(\zeta) = V_1(\zeta, z') \in L^p(\mathbb{C}),$$

and

$$\bar{\partial}_\zeta \tilde{u} = \tilde{V}\, \tilde{u} \qquad \text{in } \mathbb{C}$$

in the sense of distributions. Moreover, since V is compactly supported in $\mathbb{C}^n$, the function $\tilde{V}$ is compactly supported in $\mathbb{C}$ for almost every z'. For each fixed z', as $|\zeta| \to \infty$, $|(\zeta, z')| \to \infty$, and thus by assumption

$$\lim_{\zeta \to \infty} \tilde{u}(\zeta) = 0.$$

The one-dimensional result therefore yields $\tilde{u} \equiv 0$ for almost every $z' \in \mathbb{C}^{n-1}$. Hence, by Fubini's theorem, $u \equiv 0$ in $\mathbb{C}^n$. $\square$

5 Proof of Theorem 1.3

In this section, we prove Theorem 1.3 with L^2 potentials under L^2-flatness decay assumption. We first establish a one-dimensional unique continuation result at infinity. This is obtained by inversion from Theorem 2.1 part 2 at a finite point. To pass to higher dimensions using the coordinate slicing, we further show, using Fubini's theorem and a dyadic argument, that the global L^2-flatness of u implies L^2-flatness on almost every such slice. The one-dimensional result can then be applied on almost every slice.

Proof of Theorem 1.3: We first prove the case $n = 1$. Let $v(z) := u(\frac{1}{z})$ and $W(z) := -\frac{V(\frac{1}{z})}{\bar{z}^2}$ on $D_{\frac{1}{R}} \setminus \{0\}$. Then by Lemma 2.5, $v \in W^{1,2}_{loc}(D_{\frac{1}{R}} \setminus \{0\})$ and

$$\bar{\partial} v = W v \quad \text{on} \quad D_{\frac{1}{R}} \setminus \{0\}$$

in the sense of distributions. We next verify that v satisfies the hypothesis in Theorem 2.1 part 2.

Since $u \in L^2(\mathbb{C} \setminus \overline{D_R})$, making use of change of variables, we infer

$$\int_{D_{\frac{1}{R}}} |v(z)|^2 dv_z = \int_{\mathbb{C} \setminus \overline{D_R}} \frac{|v(\frac{1}{z})|^2}{|z|^4} dv_z = \int_{\mathbb{C} \setminus \overline{D_R}} \frac{|u(z)|^2}{|z|^4} dv_z \leq \max\{1, R^{-4}\} \int_{\mathbb{C} \setminus \overline{D_R}} |u(z)|^2 dv_z < \infty.$$

Similarly, since $V \in L^2(\mathbb{C} \setminus \overline{D_R})$ and

$$\int_{D_{\frac{1}{R}}} |W(z)|^2 dv_z = \int_{D_{\frac{1}{R}}} \frac{|V(\frac{1}{z})|^2}{|z|^4} dv_z = \int_{\mathbb{C} \setminus \overline{D_R}} |V(z)|^2 dv_z < \infty.$$

In particular, $v \in L^2(D_{\frac{1}{R}})$ and $Wv \in L^1(D_{\frac{1}{R}})$.

We then apply a removable singularity result of Harvey-Polking [9] (see also [18, Lemma A.1]) to obtain

$$\bar{\partial} v = W v \quad \text{on} \quad D_{\frac{1}{R}} \tag{5.1}$$

in the sense of distributions. Moreover, since

$$\int_{D_{\frac{1}{R}}} |\nabla v(z)|^2 dv_z = \int_{\mathbb{C}\setminus\overline{D_R}} \frac{|\nabla v(\frac{1}{z})|^2}{|z|^4} dv_z = \int_{\mathbb{C}\setminus\overline{D_R}} |\nabla u(z)|^2 dv_z < \infty,$$

it follows that

$$v \in W^{1,2}(D_{\frac{1}{R}}).$$

On the other hand, the L^2 flatness assumption of u at infinity further leads to

$$r^{-m}\int_{|z|<r} |v(z)|^2 dv_z = r^{-m}\int_{|z|>\frac{1}{r}} \frac{|u(z)|^2}{|z|^4} dv_z \le r^{4-m}\int_{|z|>\frac{1}{r}} |u(z)|^2 dv_z \to 0$$

as $r \to 0$.

Altogether we have $v \in W^{1,2}(D_{\frac{1}{R}})$ satisfying (5.1) for some $W \in L^2(D_{\frac{1}{R}})$, and v vanishes to infinite order in the L^2 sense at 0. Making use of Theorem 2.1 part 2 we have $v \equiv 0$ on $D_{\frac{1}{R}}$. Thus $u = 0$ on $\mathbb{C} \setminus \overline{D_R}$.

We next assume $n \ge 2$. Write $V = \sum_{j=1}^n V_j\, d\bar z_j$ and decompose $z = (z_1, z')$, where $z' \in \mathbb{C}^{n-1}$. For almost every fixed z' with $|z'| < R/2$, set $\tilde u(\zeta) := u(\zeta, z')$, $\tilde V(\zeta) := V_1(\zeta, z')$. By Lemma 2.6,

$$\tilde u \in W^{1,2}\left(\mathbb{C} \setminus \overline{D_{\sqrt{R^2-|z'|^2}}}\right), \qquad \tilde V \in L^2\left(\mathbb{C} \setminus \overline{D_{\sqrt{R^2-|z'|^2}}}\right)$$

for almost every such z' and satisfies

$$\bar\partial_\zeta \tilde u = \tilde V\, \tilde u \qquad \text{on} \ \ \mathbb{C} \setminus \overline{D_{\sqrt{R^2-|z'|^2}}}$$

in the sense of distributions.

We claim that, for almost every such z', $\tilde u$ vanishes to infinite order at infinity in the L^2 sense. Let

$$F(r) := \int_{|z|>r} |u(z)|^2\, dv_z$$

and

$$F_{z'}(r) := \int_{|\zeta|>r} |\tilde u(\zeta)|^2\, dv_\zeta.$$

For every $r > R$,

$$\int_{|z'|<R/2} F_{z'}(r)\, dv_{z'} \le F(r).$$

Fix a positive integer m. By (1.4), for all sufficiently large integers j,

$$F(2^j) \le 2^{-j(m+2)}.$$

Hence

$$\sum_j 2^{jm}\int_{|z'|<R/2} F_{z'}(2^j)\, dv_{z'} \le \sum_j 2^{jm} F(2^j) < \infty.$$

By Tonelli's theorem, for almost every z' with $|z'| < R/2$,

$$2^{jm} F_{z'}(2^j) \to 0 \qquad \text{as } j \to \infty.$$

Since $F_{z'}(r)$ is nonincreasing in r, it follows that

$$r^m F_{z'}(r) \to 0 \qquad \text{as } r \to \infty.$$

Taking the intersection over $m \in \mathbb{N}$, we conclude that $\tilde{u}$ is L^2-flat at infinity for almost every z'.

The one-dimensional result therefore gives $\tilde{u} \equiv 0$ for almost every z' with $|z'| < R/2$. By Fubini's theorem, u vanishes almost everywhere on a nonempty open subset of $\mathbb{C}^n \setminus \overline{B_R}$. The weak unique continuation property in Theorem 2.1 part 1, then yields $u \equiv 0$ on $\mathbb{C}^n \setminus \overline{B_R}$. □

The L^2-flatness condition in Theorem 1.3 is satisfied under a variety of natural pointwise decay assumptions, as illustrated by the following examples.

Example 3. *Suppose $u = (u_1, \ldots, u_N)^T \in W^{1,2}(\mathbb{C}^n \setminus \overline{B_R})$ satisfies $\bar{\partial} u = V u$ on $\mathbb{C}^n \setminus \overline{B_R}$, where $V \in L^2(\mathbb{C}^n \setminus \overline{B_R})$ is a matrix-valued $(0,1)$-form. Suppose that u decays faster than every algebraic rate, namely, for every $k > 0$ there exists $C_k > 0$ such that*

$$|u(z)| \le C_k |z|^{-k}, \qquad |z| \gg 1.$$

Given $m > 0$, choose

$$k > \frac{m + 2n}{2}.$$

Then

$$r^m \int_{|z|>r} |u(z)|^2 \, dv_z \lesssim r^m \int_r^\infty s^{2n-1-2k} \, ds \lesssim r^{m+2n-2k} \to 0.$$

Thus (1.4) holds, and Theorem 1.3 yields $u \equiv 0$.

In particular, this applies to solutions satisfying

$$|u(z)| \le e^{-c|z|^\alpha}, \qquad |z| \gg 1,$$

for some $c, \alpha > 0$, as well as

$$|u(z)| \le e^{-c(\ln |z|)^{1+\beta}}, \qquad |z| \gg 1,$$

for some $c, \beta > 0$, since both decay faster than every negative power of $|z|$.

The following example shows that the conclusion of Theorem 1.3 may fail for potentials outside L^2, even when the potential belongs to L^p for some $p > 2n$.

Example 4. *For each $p > 2n$, choose $\epsilon \in \left(0, \frac{p-2n}{p}\right)$, so that $(1-\epsilon)p > 2n$. Define*

$$u_\epsilon(z) := e^{-|z|^\epsilon}, \qquad z \in \mathbb{C}^n \setminus \overline{B_1}.$$

Then $u_\epsilon \in W^{1,2}(\mathbb{C}^n \setminus \overline{B_1})$ and vanishes to infinite order at infinity in the L^2 sense. Moreover, $\bar{\partial} u_\epsilon = V u_\epsilon$, where

$$V = -\frac{\epsilon}{2} |z|^{\epsilon - 2} \sum_{j=1}^n z_j \, d\bar{z}_j$$

satisfies $|V(z)| = \frac{\epsilon}{2|z|^{1-\epsilon}}$ and $V \in L^p(\mathbb{C}^n \setminus \overline{B_1})$. Thus for every $p > 2n$, there exist potentials $V \in L^p(\mathbb{C}^n \setminus \overline{B_1}) \setminus L^2(\mathbb{C}^n \setminus \overline{B_1})$ for which the conclusion of Theorem 1.3 fails.

The following example shows that unique continuation may fail for solutions with decay of finite algebraic order, and hence illustrates the essential role of the L^2-flatness assumption in Theorem 1.3.

Example 5. *For each* $\alpha \in (0, \frac{1}{2})$, *define*

$$u = \frac{1}{ze^{(\ln |z|^2)^\alpha}}.$$

Then $u \in W^{1,2}(\mathbb{C} \setminus \overline{D_2})$ *and satisfies* $\bar{\partial} u = Vu$ *on* $\mathbb{C} \setminus \overline{D_2}$, *with*

$$V = \frac{-\alpha}{\bar{z}(\ln |z|^2)^{1-\alpha}} \in L^2(\mathbb{C} \setminus \overline{D_2}).$$

Moreover, u *does not vanish to infinite order at infinity in the* L^2 *sense.*

Proof. The function u is smooth on $\mathbb{C} \setminus \overline{D_2}$. A direct computation gives

$$\bar{\partial} u = \frac{-\alpha}{|z|^2(\ln |z|^2)^{1-\alpha} e^{(\ln |z|^2)^\alpha}};$$

$$\partial u = -\frac{1}{z^2 e^{(\ln |z|^2)^\alpha}} - \frac{\alpha}{z^2(\ln |z|^2)^{1-\alpha} e^{(\ln |z|^2)^\alpha}}.$$

Hence u satisfies $\bar{\partial} u = Vu$ on $\mathbb{C} \setminus \overline{D_2}$.

We next verify the required integrability. Since for every $k \in \mathbb{Z}^+$, $e^{2(2\ln s)^\alpha} > (\ln s)^{k\alpha}$ for all sufficiently large s, we may choose $k > \frac{1}{\alpha}$ and obtain

$$\int_{|z|>2} |u|^2 dv_z \lesssim \int_2^\infty \frac{ds}{se^{2(2\ln s)^\alpha}} \lesssim \int_2^\infty \frac{ds}{s(\ln s)^{k\alpha}} \lesssim \int_{\ln 2}^\infty s^{-k\alpha} ds \lesssim (\ln 2)^{1-k\alpha} < \infty.$$

On the other hand, using the facts that $e^{2(2\ln s)^\alpha} > 1$, $\ln s > 1$ for all sufficiently large s and $0 < \alpha < \frac{1}{2}$,

$$\int_{|z|>2} |\nabla u|^2 dv_z \lesssim \int_{|z|>2} \frac{1}{|z|^4} dv_z \lesssim \int_2^\infty s^{-3} ds < \infty.$$

Similarly,

$$\int_{\mathbb{C}\setminus\overline{D_2}} |V|^2 dv_z \lesssim \int_2^\infty \frac{1}{s(\ln s)^{2-2\alpha}} ds = \int_{\ln 2}^\infty s^{2\alpha-2} ds \lesssim (\ln 2)^{2\alpha-1} < \infty.$$

Together, we have $u \in W^{1,2}(\mathbb{C} \setminus \overline{D_2})$ and $V \in L^2(\mathbb{C} \setminus \overline{D_2})$.

It remains to show that u does not vanish to infinite order at infinity in the L^2 sense. When r is sufficiently small, we have

$$\int_{|z|>\frac{1}{r}} |u|^2 dv_z \gtrsim \int_{\frac{1}{r}}^\infty \frac{1}{se^{2(2\ln s)^\alpha}} ds.$$

Since $0 < \alpha < \frac{1}{2}$, for all sufficiently large s, $e^{2(2\ln s)^\alpha} < s$. Therefore, for $r > 0$ sufficiently small,

$$\int_{|z|>\frac{1}{r}} |u|^2 dv_z \gtrsim \int_{\frac{1}{r}}^\infty \frac{1}{s^2} ds \gtrsim r.$$

In particular, taking any $m > 1$, we get

$$r^{-m} \int_{|z|>\frac{1}{r}} |u|^2 \, dv_z \gtrsim r^{1-m} \to \infty \quad \text{as } r \to 0.$$

The proof is complete. $\square$

Despite the failure of the unique continuation property in Theorem 1.3 when $V \notin L^2$ near infinity as demonstrated above, unique continuation may still hold under additional structural assumption on the potential, for instance when the potential $V = O(\frac{1}{|z|})$ at infinity. Such condition allows potentials outside L^2; indeed, the model case $\frac{1}{|z|}$ belong to $L^p \setminus L^2$ for all $2n < p < \infty$ on $\mathbb{C}^n \setminus \overline{B_R}$.

Theorem 5.1. *Suppose $u = (u_1, \dots, u_N)^T \in W^{1,2}(\mathbb{C}^n \setminus \overline{B_R})$ satisfies*

$$\bar{\partial} u = Vu \quad \text{on} \quad \mathbb{C}^n \setminus \overline{B_R}$$

for some measurable matrix-valued $(0,1)$-form V on $\mathbb{C}^n \setminus \overline{B_R}$ with $|V| \le \frac{C}{|z|}$ for some $C > 0$. Assume that u vanishes to infinite order at infinity in the L^2 sense. Then u vanishes identically if either $N = 1$, or if $N \ge 2$ and $C < \frac{1}{4}$.

Proof. When $n = 1$, define $w(z) := u(\frac{1}{z})$ and $W(z) := -\frac{V(\frac{1}{z})}{\bar{z}^2}$ on $D_{\frac{1}{R}} \setminus \{0\}$. As in the proof of Theorem 1.3, $w \in W^{1,2}(D_{\frac{1}{R}})$ and vanishes to infinite order at 0. Moreover, $\bar{\partial} w = Ww$ with $|W(z)| = \frac{|V(\frac{1}{z})|}{|z|^2} \le \frac{C}{|z|}$ on $D_{\frac{1}{R}}$. Making use of Theorem 2.1 part 3, we have $w \equiv 0$ on $D_{\frac{1}{R}}$. Thus $u = 0$ on $\mathbb{C} \setminus \overline{D_R}$.

The case $n \ge 2$ is reduced to the one-dimensional result by the same slicing argument as in the proof of Theorem 1.3. Write $V = \sum_{j=1}^n V_j \, d\bar{z}_j$ and decompose $z = (z_1, z')$, where $z' \in \mathbb{C}^{n-1}$. For almost every fixed z' with $|z'| < R/2$, the slice $\tilde{u}(\zeta) := u(\zeta, z')$ belongs to $W^{1,2}\left(\mathbb{C} \setminus \overline{D_{\sqrt{R^2 - |z'|^2}}}\right)$ and satisfies

$$\bar{\partial}_\zeta \tilde{u} = \tilde{V}\, \tilde{u}, \qquad \tilde{V}(\zeta) := V_1(\zeta, z'),$$

with

$$|\tilde{V}(\zeta)| \le \frac{C}{\sqrt{|\zeta|^2 + |z'|^2}} \le \frac{C}{|\zeta|}.$$

As in the proof of Theorem 1.3, $\tilde{u}$ is L^2-flat at infinity for almost every such z'. The one-dimensional result therefore gives $\tilde{u} \equiv 0$ for almost every z' with $|z'| < R/2$. By Fubini's theorem, u vanishes on a nonempty open subset of $\mathbb{C}^n \setminus \overline{B_R}$, and Theorem 2.1, part 1, then yields $u \equiv 0$ on $\mathbb{C}^n \setminus \overline{B_R}$. □

As an immediate consequence of Theorem 1.3, we obtain the following unique continuation result at infinity for the full gradient operator.

Corollary 5.2. *Suppose $u = (u_1, \dots, u_N)^T \in W^{1,2}(\mathbb{R}^{2n} \setminus \overline{B_R})$ satisfies $|\nabla u| \le V|u|$ for some nonnegative $V \in L^2(\mathbb{R}^{2n} \setminus \overline{B_R})$. If u vanishes to infinite order at infinity in the L^2 sense, then u vanishes identically.*

Proof. Identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$. Then

$$|\bar{\partial} u| \le |\nabla u| \le V|u|.$$

The conclusion now follows from Theorem 1.3. □

Theorem 1.3 can also be applied to study the uniqueness of certain nonlinear PDEs concerning Laplacian under sufficiently rapid decay at infinity.

Corollary 5.3. *Let $\psi = (\psi_1, \ldots, \psi_N)^T \in L^1_{loc}((\mathbb{R}^2 \setminus \overline{D_R}) \times \mathbb{R}^{2N})$ satisfy*

$$|\psi(x, y_1) - \psi(x, y_2)| \leq V(x)|y_1 - y_2|, \quad \text{for a.e. } \ x \in \mathbb{R}^2 \setminus \overline{D_R}, \ y_1, y_2 \in \mathbb{R}^{2N}$$

for some nonnegative $V \in L^2(\mathbb{R}^2 \setminus \overline{D_R})$. Then there exists at most one $W^{2,2}(\mathbb{R}^2 \setminus \overline{D_R})$ real-valued vector solution $u = (u_1, \ldots, u_N)^T$ to

$$\Delta u = \psi(\cdot, \nabla u) \quad \text{on} \ \ \mathbb{R}^2 \setminus \overline{D_R}$$

such that ∇u vanishes to infinite order at infinity in the L^2 sense.

Proof. Suppose that u_1 and u_2 both satisfy $\Delta u = \psi(x, \nabla u)$ on $\mathbb{R}^2 \setminus \overline{D_R}$, and $\nabla u_j(x)$ vanishes to infinite order at infinity in the L^2 sense, $j = 1, 2$. Let $v := u_1 - u_2$. Then

$$|\Delta v(x)| = |\psi(x, \nabla u_1(x)) - \psi(x, \nabla u_2(x))| \leq V(x)|\nabla v(x)| \ \text{ on } \ \mathbb{R}^2 \setminus \overline{D_R},$$

and ∇v is L^2-flat at infinity.

With the convention $\bar\partial = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$, we have $\Delta v(x) = 4\bar\partial\partial v(z)$. Since v is real-valued componentwise, $|\nabla v(x)| = 2|\partial v(z)|$. Let

$$\tilde{v} := \partial v.$$

Then $\tilde{v} \in W^{1,2}(\mathbb{R}^2 \setminus \overline{D_R})$, and the preceding inequality implies

$$|\bar\partial \tilde{v}| \leq \frac{1}{2} V |\tilde{v}| \ \text{ on } \ \mathbb{R}^2 \setminus \overline{D_R}$$

Moreover, since $|\tilde{v}| = \frac{1}{2}|\nabla v|$, $\tilde{v}$ is L^2-flat at infinity. Applying Theorem 1.3, we obtain $\tilde{v} \equiv 0$. Since v is real-valued and $v \in L^2(\mathbb{R}^2 \setminus \overline{D_R})$, this implies that $v \equiv 0$. ☐

References

[1] S. Alinhac and S. Baouendi: *A counterexample to strong uniqueness for partial differential equations of Schrödinger's type.* Comm. Partial Differential Equations **19**(1994), no. 9-10, 1727–1733. 5

[2] J. Bourgain and C. E. Kenig: *On localization in the continuous Anderson-Bernoulli model in higher dimension.* Invent. Math. **161**(2005), no.2, 389–426. 1, 2, 8, 11

[3] W. Chen, Y. Fan and X. Tang: *On a sharp constant of unique continuation for $\bar\partial$.* In preparation. 5

[4] E. Chirka and J.-P. Rosay: *Remarks on the proof of a generalized Hartogs lemma.* Ann. Polon. Math. **70**(1998), 43–47. 3, 20

[5] J.-P. Demailly: *Estimations L^2 pour l'opérateur $\bar\partial$ d'un fibré vectoriel holomorphe semi-positif au-dessus d'une variété kählérienne complète.* Ann. Sci. École Norm. Sup. (4) **15**(1982), no. 3, 457–511. 9

[6] R. Frank and P. Ivanisvili: *Counterexamples to the Landis conjecture in dimensions three and higher.* Preprint. arXiv:2608.00802. 1

[7] X. Gong and J.-P. Rosay: *Differential inequalities of continuous functions and removing singularities of Rado type for J-holomorphic maps.* Math. Scand. **101**(2007), no. 2, 293–319. 2, 9

[8] L. Hörmander: *Notions of convexity.* Birkhäuser Boston, Inc., Boston, MA, 2007, viii+414 pp.

[9] R. Harvey and J. Polking: *Removable singularities of solutions of linear partial differential equations.* Acta Math. **125**(1970), 39–56. 21

[10] C. Kenig: *Some recent quantitative unique continuation theorems.* Séminaire: Équations aux Dérivées Partielles. 2005–2006, Sémin. Équ. Dériv. Partielles, École Polytech., Palaiseau, 2006, pp. Exp. No. XX, 12. 1

[11] C. Kenig, L. Silvestre and J.-N. Wang: *On Landis' conjecture in the plane.* Comm. Partial Differential Equations, **40**(2015), no. 4, 766–789. 2, 8

[12] V. A. Kondratiev and E. M. Landis: *Qualitative theory of second-order linear partial differential equations.* Itogi Nauki i Tekhniki Sovrem. Probl. Mat. Fund. Naprav., 32, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1988, 99–215. 1

[13] I. Lieb and J. Michel: *The Cauchy-Riemann complex.* Aspects Math., E34 Friedr. Vieweg & Sohn, Braunschweig, 2002, x+362 pp. 6, 10

[14] A. Logunov, E. Malinnikova, N. Nadirashvili and F. Nazarov: *The Landis conjecture on exponential decay.* Invent. Math. **241**(2025), no. 2, 465–508. 1

[15] N. Mandache: *A counterexample to unique continuation in dimension two.* Comm. Anal. Geom. **10**(2002), no. 1, 1–10. 5

[16] V. Z. Meshkov; *On the possible rate of decrease at infinity of the solutions of second-order partial differential equations.* Mat. Sb. **182**(1991), no. 3, 364–383. Math. USSR-Sb. **72**(1992), no. 2, 343–361. 1, 4

[17] Y. Pan and T. Wolff: *A remark on unique continuation.* J. Geom. Anal. **8**(1998), no. 4, 599–604. 5

[18] Y. Pan and Y. Zhang: *Unique continuation for $\bar{\partial}$ with square-integrable potentials.* New York Journal of Mathematics, **29**(2023), 402–416. 4, 5, 21

[19] Y. Pan and Y. Zhang: *Weighted Sobolev estimates of the truncated Beurling operator.* Bull. Lond. Math. Soc., **56**(2024), no. 1, 232–239. 18

[20] Y. Pan and Y. Zhang: *Unique continuation of Schrödinger-type equations for $\bar{\partial}$.* Comm. Anal. Geom. **33**(2025), no. 8, 1975–2014. 5

[21] Y. Pan and Y. Zhang: *A remark on unique continuation of $\bar{\partial}$.* Nagoya Math. J., to appear. 3, 4, 5

[22] Z. Shi: *A unique continuation property for $|\bar{\partial}u| \leq V|u|$.* Math. Scand. **131**(2025), no. 1, 113–154. 5

[23] I. N. Vekua: *Generalized analytic functions*, International Series of Monographs on Pure and Applied Mathematics, Vol. 25, Pergamon Press, 1962 xxix+668 pp. 18

[24] W. Ziemer: *Weakly differentiable functions. Sobolev spaces and functions of bounded variation.* Graduate Texts in Mathematics, 120. Springer-Verlag, New York, 1989. xvi+308 pp. 6, 7, 9

pan1@pfw.edu,

Department of Mathematical Sciences, Purdue University Fort Wayne, Fort Wayne, IN 46805-1499, USA.

zhan1313@pfw.edu,

Department of Mathematical Sciences, Purdue University Fort Wayne, Fort Wayne, IN 46805-1499, USA.